

On The b -Chromatic Number of Regular Bounded Graphs

El Sahili Amine / Kouider Mekkia

Mortada Maidoun

Abstract

A b -coloring of a graph is a proper coloring such that every color class contains a vertex adjacent to at least one vertex in each of the other color classes. The b -chromatic number of a graph G , denoted by $b(G)$, is the maximum integer k such that G admits a b -coloring with k colors. El Sahili and Kouider conjectured that $b(G) = d + 1$ for d -regular graph with girth 5, $d \geq 4$. In this paper, we prove that this conjecture holds for d -regular graph with at least $d^3 + d$ vertices. More precisely we show that $b(G) = d+1$ for d -regular graph with at least $d^3 + d$ vertices and containing no cycle of order 4. We also prove that $b(G) = d + 1$ for d -regular graphs with at least $2d^3 + 2d - 2d^2$ vertices improving Cabello and Jakovac bound.

Keywords: proper coloring, b -coloring, b -chromatic number.

1 Introduction

A proper coloring of a graph $G = (V; E)$ is an assignment of colors to the vertices of G , such that any two adjacent vertices have different colors. The chromatic number of G , denoted by $\chi(G)$ is the smallest integer k such that G has a proper coloring with k colors. A color class in a proper coloring of a graph G is the subset of V containing all the vertices of same color. A proper coloring of a graph is called a b -coloring, if each color class contains a vertex adjacent to at least one vertex of each of the other color classes. Such a vertex is called a dominant vertex. The b -chromatic number of a graph G , denoted by $b(G)$, is the largest integer k such that G has a b -coloring with k colors. For a given graph G , it may be easily remarked that $\chi(G) \leq b(G) \leq \Delta(G) + 1$.

The b -chromatic number of a graph was introduced by Irving and Manlove [10] when considering minimal proper colorings with respect to a partial order defined on the set of all partitions of the vertices of a graph. They proved

that determining $b(G)$ is NP-hard for general graphs, but polynomial-time solvable for trees.

Recently, Kratochvil *et al.*[11] have shown that determining $b(G)$ is NP-hard even for bipartite graphs while Corteel, Valencia-Pabon, and Vera [5] proved that there is no constant $\epsilon > 0$ for which the b -chromatic number can be approximated within a factor of $120/133-\epsilon$ in polynomial time (*unless* $P = NP$).

Finally, Balakrishnan and Francis Raj [1, 2] investigated the b -chromatic number of the Mycielskians and vertex deleted subgraphs. Hoang and Kouider [9] characterize all bipartite graphs G and all P_4 -sparse graphs G such that each induced subgraph H of G satisfies $b(H) = \chi(H)$. In [8], Effantin and Kheddouci gave the exact value for the b -chromatic number of power graphs of a path and determined bounds for the b -chromatic number of power graphs of a cycle.

In [6], Kouider and El-Sahili formulated the following conjecture: For a d -regular graph with girth 5, $b(G) = d + 1$ for $d \geq 4$. They proved that for every graph G with girth at least 6, $b(G)$ is at least the minimum degree of the graph, and if this graph is d -regular then $b(G) = d + 1$. This conjecture have been proved in [6] for the particular case when G contains no cycles of order 6. In [3], Maffray *et.al* proved that the conjecture holds for d -regular graphs different from the Petersen graph and $d \leq 6$. The f -chromatic vertex number of a d -regular graph G , denoted by $f(G)$, is the maximum number of dominant vertices with distinct color classes in a proper $d + 1$ -coloring of G . In [7], El Sahili *et al.* proved that $f(G) \leq b(G)$ and then reformulated El Sahili and Kouider conjecture as follows: $f(G) = d + 1$ for d -regular graph with no cycle of order 4. They proved that (i) $f(G) \geq \lfloor \frac{d+1}{2} \rfloor + 2$ for a d -regular graph containing no cycle of order 4; (ii) $f(G) \geq \lceil \frac{d+1}{2} \rceil + 4$ for d -regular graph containing no cycle of order 4 and of diameter 5; (iii) $b(G) = d + 1$ for a d -regular graph containing no cycle of order 4 nor of order 6; (v) $b(G) = d + 1$ for a d -regular graph with no cycle of order 4 and of diameter at least 6. In this paper, we prove in two different methods that El Sahili and Kouider conjecture holds for a d -regular graph containing at least $d^3 + d$ vertices and more precisely we prove that $b(G) = d + 1$ for a d -regular graph containing no cycle of order 4 and at least $d^3 + d$ vertices.

In [11], Kratochvil *et al.* proved that for a d -regular graph G with at least d^4 vertices, $b(G) = d+1$. It follows from their result that for any d , there is only a finite number of d -regular graphs G with $b(G) \leq d$. In [4], using matchings, Cabello and Jakovac reduced the bound of d^4 vertices to $2d^3 - d^2 + d$ vertices. In this paper, we show that, without using matching, $b(G) = d+1$ for a d -regular graphs with $v(G) \geq 2d^3 + 2d - 2d^2$ improving Cabello and Jakovac bound.

2 Lower Bounded Graphs

Consider a d -regular graph G and let K and F be 2 disjoint and fixed induced subgraphs of G . Suppose that the vertices of K are colored by a proper $d+1$ -coloring. Also, suppose that the vertices of F are colored by a proper $d+1$ -coloring c . We define a digraph Δ_c where $V(\Delta_c) = \{1, 2, \dots, d+1\}$ and $E(\Delta_c) = \{(i, j) : \text{a vertex of color } i \text{ in } F \text{ is not adjacent to a vertex of color } j \text{ in } K\}$. Δ_c is called a coloring digraph. Note that the coloring digraph Δ_c may contain loops and circuits of length 2. The number of loops in Δ_c is denoted by $\ell(\Delta_c)$. We introduce the following lemma:

Lemma 2.1. *If $(i, i) \notin E(\Delta_c)$ and if there exists a circuit C in Δ_c containing i , then we can recolor $V(F)$ by a proper $d+1$ -coloring c' such that $\ell(\Delta_{c'}) > \ell(\Delta_c)$.*

Proof. Suppose that $(i, i) \notin E(\Delta_c)$ and there exists a circuit C in Δ_c containing i . Without loss of generality, suppose that $C = 1 \ 2 \ \dots \ i$. We define a new proper coloring c' , where for $v \in F$

$$c'(v) = \begin{cases} c(v) & \text{if } c(v) \notin C \\ c(v) + 1 & \text{if } c(v) \in C \setminus \{i\} \\ 1 & \text{if } c(v) = i \end{cases}$$

A loop (s, s) in Δ_c is clearly a loop in $\Delta_{c'}$ whenever $s \geq i+1$. Since $(s, s+1)$ and $(i, 1) \in E(\Delta_c)$, $1 \leq s \leq i-1$, then (l, l) is a loop in $\Delta_{c'}$, $\forall l \ 1 \leq l \leq i$. Thus, $\ell(\Delta_{c'}) > \ell(\Delta_c)$. \square

Theorem 2.1. *Let G be a d -regular graph with no cycle of order 4. If $v(G) \geq d^3 + d$, then $b(G) = d+1$*

Proof. Suppose that k vertices and their neighbors are colored by a proper $d + 1$ -coloring in such a way that these k vertices are dominant of color $1, 2, \dots, k$, $k \leq d$. Let C be the set of colored vertices, then $|C| = k + kd \leq d(d + 1)$.

Let $R_i = \{v \in R : v \text{ has exactly } i \text{ neighbors in } C\}$, $0 \leq i \leq d$, and set $R = N(C) = \bigcup_{i=1}^d R_i$.

Let $R_a = R_2 \cup \dots \cup R_{\lfloor \frac{d+1}{2} \rfloor}$, $R_b = R_4 \cup \dots \cup R_{\lfloor \frac{d+1}{2} \rfloor}$, and $R_c = R_{\lfloor \frac{d+1}{2} \rfloor + 1} \cup \dots \cup R_d$. Since dominant vertices has no neighbors in $V(G) \setminus C$ and a neighbor of a dominant vertex has at most $d - 1$ neighbors in R , then by double counting the edges between C and $(R_1 \cup R_2 \cup \dots \cup R_d)$ we can say that:

$$|R| + |R_a| + 2|R_b| + \lfloor \frac{d+1}{2} \rfloor |R_c| \leq d^2(d - 1) \quad (a)$$

Let

$$S_1 = \{v \notin C \cup R : |N(v) \cap R_a| \geq d - 2\}$$

$$S_2 = \{v \notin C \cup R : |N(v) \cap R_b| \geq \lceil \frac{d-1}{2} \rceil\}$$

$$S_3 = \{v \notin C \cup R : |N(v) \cap R_c| \geq 1\}$$

We have

$$(d - 2)|S_1| \leq (d - 2)|R_a|, \text{ then } |S_1| \leq |R_a|$$

$$\lceil \frac{d-1}{2} \rceil |S_2| \leq (d - 4)|R_b|, \text{ then } |S_2| < 2|R_b|$$

$$\text{and } |S_3| \leq (\lceil \frac{d-1}{2} \rceil - 1)|R_c|$$

So,

$$|C| + |R| + |S_1| + |S_2| + |S_3| < |C| + |R| + |R_a| + 2|R_b| + (\lceil \frac{d-1}{2} \rceil - 1)|R_c|$$

But, by (a), we have $\lfloor \frac{d+1}{2} \rfloor |R_c| \leq d^2(d - 1) - |R| - |R_a| - 2|R_b|$, thus

$$|C| + |R| + |S_1| + |S_2| + |S_3| < |C| + |R| + |R_a| + 2|R_b| + d^2(d - 1) - |R| - |R_a| - 2|R_b| \leq d(d + 1) + d^2(d - 1) \leq d^3 + d$$

Since $v(G) \geq d^3 + d$, then there exists a vertex y such that $y \notin C \cup R \cup S_1 \cup S_2 \cup S_3$. We note that

$$|N(y) \cap R_a| \leq d - 3, \quad |N(y) \cap R_b| \leq \lceil \frac{d-1}{2} \rceil - 1, \quad |N(y) \cap R_c| = 0.$$

Thus, we have

$$|N(y) \cap (R_0 \cup R_1)| \geq 3 \quad (b).$$

Let K and F be two induced subgraphs of G where $V(K) = C$ and $V(F) = N(y) \cup \{y\}$. Color y and its neighbors by a proper $d+1$ -coloring c in such a way that y is a dominant vertex of color $k+1$ and $\ell(\Delta_c)$ is maximal. If $\ell(\Delta_c) = d+1$, then y is a dominant vertex in a proper $d+1$ -coloring for $V(K) \cup V(F)$. Else, there exists $i \neq k+1$ such that $(i, i) \notin E(\Delta_c)$. Let x be the vertex of color i in $N(y)$.

There exists at least a neighbor of y , say w_1 , such that w_1 has no neighbor of color i in K since G has no C_4 and $k \leq d$. Also, by (b), there exist at least three neighbors of y , say w_2, w_3 and w_4 , such that each of them has at most one neighbor in K . Since x has at most $\lfloor \frac{d+1}{2} \rfloor$ neighbors in K and $y \notin C \cup R \cup S_1 \cup S_2 \cup S_3$, then there exists a neighbor of y , say w_5 , such that x has no neighbor of color $c(w_5)$ ($c(w_5) \neq k+1$) in K and w_5 has at most 3 neighbors in K . If x has no neighbor of color $c(w_1)$ in K , then $C_1 = i \ c(w_1)$ is a circuit in Δ_c . Else, i has a neighbor in K of color $c(w_1)$. If there exists j , $2 \leq j \leq 4$, such that x has no neighbor of color $c(w_j)$ in K , then C_2 or C_3 is a circuit in Δ_c , where $C_2 = i \ c(w_j)$ and $C_3 = i \ c(w_j) \ c(w_1)$, since w_j has at most one neighbor in K . Otherwise, neither x nor w_5 belongs to the set $\{w_1, w_2, w_3, w_4\}$ since x has no neighbor in K of color $c(w_5)$ and it has more than one neighbor. If w_5 is not adjacent to a vertex of color i in K , then $C_3 = i \ c(w_5)$ is a circuit in Δ_c . Otherwise, there exists j , $1 \leq j \leq 4$, such that w_5 has no neighbor of color $c(w_j)$ in K since w_5 has at most 3 neighbors in K . If $j=1$, then $C_4 = c(w_5) \ c(w_1) \ i$ is a circuit in Δ_c . Else, C_5 or C_6 is a circuit in Δ_c , where $C_5 = c(w_5) \ c(w_j) \ i$ and $C_6 = c(w_5) \ c(w_j) \ c(w_1) \ i$, since w_j has at most one neighbor in K . In all cases, there exists a circuit containing i . Then, by Lemma 2.1 we can find a proper $d+1$ -coloring c' of $V(F)$ such that $\ell(\Delta_{c'}) > \ell(\Delta_c)$, a contradiction. Thus, $\ell(\Delta_c) = d+1$ and so c is a proper $d+1$ -coloring for $V(F) \cup V(K)$ and y is a dominant vertex of color $k+1$. This proves that we can find a proper $d+1$ -coloring of G that contains $d+1$ dominant vertices of distinct colors \square

The coloring digraph, which is used to prove Theorem 2.1, can be used also to establish the following result improving Cabello and Jakovac bound:

Theorem 2.2. *Let G be a d -regular graph such that $v(G) \geq 2d^3 + 2d - 2d^2$, then $b(G) = d+1$.*

Proof. Suppose that k vertices and their neighbors are colored by a proper $d+1$ -coloring in such a way that these k vertices are dominant of color $1, 2, \dots, k$, $k \leq d$. Define C , R and R_i , $0 \leq i \leq d$, as in the proof of Theorem

2.1. Let $C_i = \{v \in R : v \text{ has a neighbor of color } i \text{ in } C\}$, $1 \leq i \leq d+1$. Let $R_a = R_3 \cup \dots \cup R_{\lfloor \frac{d+1}{2} \rfloor}$ and $R_b = R_{\lfloor \frac{d+1}{2} \rfloor + 1} \cup \dots \cup R_d$.

Dominant vertices has no neighbors in $V(G) \setminus C$ and a neighbor of a dominant vertex has at most $d-1$ neighbors in R , so we can say that:

$$|R| + 2|R_a| + \lfloor \frac{d+1}{2} \rfloor |R_b| \leq d^2(d-1)$$

For $k < d$, we have

$$|C_i| \leq k(d-1) \leq (d-1)^2 \quad 1 \leq i \leq d+1$$

For $k = d$, since only $d-1$ vertices of color i , $i \neq d+1$, can have neighbors outside C while d vertices of color $d+1$ can have neighbors outside C , then

$$|C_i| \leq (d-1)^2 \text{ for } i \neq d+1, \text{ and } |C_{d+1}| \leq d(d-1)$$

Let

$$S_1 = \{v \notin C \cup R : |N(v) \cap R_a| \geq \lceil \frac{d-1}{2} \rceil\}$$

$$S_2 = \{v \notin C \cup R : |N(v) \cap R_b| \geq 1\}$$

$$S'_i = \{v \notin C \cup R : |N(v) \cap C_i| \geq d-1\}, \quad 1 \leq i \leq d+1 \text{ such that } i \neq k+1.$$

The union of the sets S'_i , $1 \leq i \leq d+1$ and $i \neq k+1$, is denoted by S' . We have

$$\lceil \frac{d-1}{2} \rceil |S_1| \leq (d-3)|R_a|, \text{ thus } |S_1| < 2|R_a|$$

$$|S_2| \leq (\lceil \frac{d-1}{2} \rceil - 1)|R_b|$$

$$(d-1)|S'_i| \leq (d-1)|C_i|, \text{ thus } |S'_i| \leq (d-1)^2$$

$$\text{and } |S'| \leq d(d-1)^2.$$

So,

$$|C| + |R| + |S_1| + |S_2| + |S'| < |C| + |R| + 2|R_a| + (\lceil \frac{d-1}{2} \rceil - 1)|R_b| + d(d-1)^2$$

But $\lfloor \frac{d+1}{2} \rfloor |R_b| \leq d^2(d-1) - |R| - 2|R_a|$, thus

$$|C| + |R| + |S_1| + |S_2| + |S'| < d(d+1) + d^2(d-1) + d(d-1)^2 \leq 2d^3 + 2d - 2d^2$$

Since $v(G) \geq 2d^3 + 2d - 2d^2$, then there exists a vertex y such that $y \notin C \cup R \cup S_1 \cup S_2 \cup S'$. We note that

$$|N(y) \cap R_a| \leq \lceil \frac{d-1}{2} \rceil - 1, |N(y) \cap R_b| = 0, |N(y) \cap C_i| \leq d-2, \\ \forall i \neq k+1 \quad (*).$$

Let K and F be two induced subgraphs where $V(K) = C$ and $V(F) = N(y) \cup \{y\}$. Color y and its neighbors by a proper $d+1$ -coloring c in such a way that y is a dominant vertex of color $k+1$ and $\ell(\Delta_c)$ is maximal. If $\ell(\Delta_c) = d+1$, then y is a dominant vertex in a proper $d+1$ -coloring for $V(K) \cup V(F)$. Else, there exists $i \neq k+1$, such that $(i, i) \notin E(\Delta_c)$. Let x be the vertex of color i in $N(y)$.

By (*), we can find at least 2 neighbors of y , say w_1 and w_2 , such that w_1 and w_2 has no neighbor of color i in K . If x has no neighbor of color $c(w_j)$, $j \in \{1, 2\}$, then $C_1 = i \ c(w_j)$ is a circuit in Δ_c . Otherwise, since x has at most $\lfloor \frac{d+1}{2} \rfloor$ neighbors in K , then there exists a neighbor of y , say w_3 , such that x has no neighbor of color $c(w_3)$ in K where $c(w_3) \neq k+1$, $w_3 \notin \{w_1, w_2\}$ and w_3 has at most 2 neighbors in K . If w_3 has no neighbor of color i in K , then $C_2 = c(w_3) \ i$ is a circuit in Δ_c . Else, w_3 has no neighbor in K of color $c(w_k)$ where $k = 1$ or 2 . So, $C_3 = c(w_3) \ c(w_k) \ i$ is a circuit in Δ_c . In all cases, there exists a circuit containing i , then by Lemma 2.1 we can find a proper $d+1$ -coloring c' of $V(F)$ such that $\ell(\Delta_{c'}) > \ell(\Delta_c)$, a contradiction. Thus, $\ell(\Delta_c) = d+1$ and so c is a proper $d+1$ -coloring for $V(F) \cup V(K)$ and y is a dominant vertex of color $k+1$. Consequently, we can find a $d+1$ dominant vertices of distinct colors. \square

3 Matching and b -coloring

Using matching Cabello and Jacovac proved that $b(G) = d+1$ for any d -regular graph with at least $2d^3 + d - d^2$ vertices. Matching also yields another proof for Theorem 2.1. This proof is based on the following Lemma:

Lemma 3.1. *Let t be a fixed integer. Let L and V be two sets of cardinality t . Let H be a bipartite graph with partition V and L such that for every $v \in V$ and every $u \in L$, $d_H(v) + d_H(u) \geq t$. Then H has a perfect matching.*

Proof. The proof is by contradiction. Let M be the maximum matching such that M is not perfect. Then, there exist at least two vertices, say $u \in L$ and $v \in V$, outside M . Since $d_H(v) + d_H(u) \geq t$, then there exists an edge in M , say ab , such that $a \in N_H(u)$ and $b \in N_H(v)$. Then let M' be the set of edges such that $M' = (M \setminus \{ab\}) \cup \{au, bv\}$. It is clear that M' is a matching with $|M'| > |M|$, a contradiction. \square

Another Proof of Theorem 2.1.

We have a partial b coloring of the graph G , the set of k dominant vertices of the colors $1, 2, \dots, k$ and their neighbors. C , R and R_i , $0 \leq i \leq d$, are defined as in the previous proof. Let $C_i = \{v \in C : v \text{ is of color } i\}$, $1 \leq i \leq d+1$. Let $R_a = R_1 \cup R_2 \cup \dots \cup R_{\lfloor \frac{d+1}{2} \rfloor}$, $R_b = R_3 \cup R_4 \cup \dots \cup R_{\lfloor \frac{d+1}{2} \rfloor}$, and $R_c = R_{\lfloor \frac{d+1}{2} \rfloor + 1} \cup \dots \cup R_d$.

Dominant vertices has no neighbors in $V(G) \setminus C$ and a neighbor of a dominant vertex has at most $d-1$ neighbors in R , so we can say that:

$$|R| + |R_2| + 2|R_b| + \lfloor \frac{d+1}{2} \rfloor |R_c| \leq d^2(d-1) \quad (a)$$

For each $y \in G - (C \cup R)$, let us set:

$$R_c(v) = N(v) \cap R_c,$$

$$R_b(v) = N(v) \cap R_b,$$

$$R'_b(v) = N(v) \cap (R_0 \cup R_1 \cup R_2),$$

$$S_1 = \{v \notin (C \cup R), |R_b(v)| \geq \lceil \frac{d-1}{2} \rceil\}.$$

$$S_2 = \{v \notin (C \cup R), |R_c(v)| \geq 1\}.$$

Then, we have

$$\lceil \frac{d-1}{2} \rceil |S_1| \leq (d-3)|R_b|, \text{ then } |S_1| \leq 2|R_b|$$

$$\text{and } |S_2| \leq (\lceil \frac{d-1}{2} \rceil - 1)|R_c|$$

Let $S_0 = \{y \in G \setminus (C \cup R \cup S_1 \cup S_2) : N(v) \cap (R_2 \cup \dots \cup R_{\lfloor \frac{d+1}{2} \rfloor + 1}) \geq d-2\}$ and $S_{0,i} = \{v \in S_0 : R_b(v) = i\}$, $0 \leq i \leq \lceil \frac{d-1}{2} \rceil - 1$. So, $S_0 = \cup_{0 \leq i \leq \lceil \frac{d-1}{2} \rceil - 1} S_{0,i}$. We have:

$$\sum_{0 \leq i \leq \lceil \frac{d-1}{2} \rceil - 1} i \cdot |S_{0,i}| + \lceil \frac{d-1}{2} \rceil |S_1| \leq (d-3) \cdot |R_b| \quad (1)$$

$$\sum_{0 \leq i \leq (\lceil \frac{d-1}{2} \rceil - 1)} (d-2-i) \cdot |S_{0,i}| \leq (d-2) \cdot |R_2| \quad (2)$$

From these 2 last inequalities, we deduce that

$$(d-2) \cdot |S_0| + 2 \lceil \frac{d-1}{2} \rceil |S_1| \leq 2(d-3)|R_b| + (d-2)|R_2|$$

Thus, we get

$$|S_0| + |S_1| < 2 \cdot |R_b| + |R_2| \quad (b)$$

Thus, by (a) and (b), we get:

$$|C| + |R| + |S_0| + |S_1| + |S_2| < |C| + |R| + 2|R_b| + |R_2| + (\lceil \frac{d-1}{2} \rceil - 1)|R_c| \leq d^2(d-1) + d(d+1) \leq d^3 + d.$$

Since, $|V(G)| \geq (d^3 + d)$, then we can find a vertex y such that $y \in G - (C \cup R \cup S_0 \cup S_1 \cup S_2)$. Color y by $k+1$. Now, we color separately $R_b(y)$ and $R'_b(y)$. Let B be a subset of $N(y)$. For any color j , let $e(C_j, B)$ be the number of edges with one extremity in C_j and the other one in B . We remark that for any color j

$$e(C_j, N(y)) \leq d-1 \quad (*)$$

By definition of y , $|R_c(y)| = 0$, $|R_b(y)| \leq \lceil \frac{d-1}{2} \rceil - 1$ and $|N(y) \cap (R - R_1)| \leq (d-3)$, so

$$|N(y) \cap (R_0 \cup R_1)| \geq 3 \quad (**)$$

Let us note that

$$e(C, R'_b(y)) = |N(y) \cap R_1| + 2|N(y) \cap R_2|$$

Let 1 be a color such that $e(C_1, R'_b(y))$ is maximum.

(1) If $e(C_1, R'_b(y)) = |R'_b(y) \setminus R_0|$, by (*) we can choose one vertex $w_1 \notin R'_b(y)$ not neighbor of c_1 and we color it by c_1 .

We have: $e(C \setminus C_1, R'_b(y)) = |R'_b(y) \cap R_2| \leq |R'_b(y)| - 3$ by inequality (**).

(2) If $e(C_1, R'_b(y)) = |R'_b(y) \setminus R_0| - 1$, we choose one vertex $w_1 \in R'_b(y)$ not neighbor of c_1 and we color it by c_1 . Now, by (**), $e(C - C_1, R'_b(y) - w_1) =$

$$|R'_b(y) \cap R_2| \leq |R'_b(y) \setminus \{w_1\}| - 2$$

(3) If (1) and (2) are excluded, for any color j , $e(C_j, R'_b(y)) \leq |R'_b(y) \setminus R_0| - 2$.

Now, we are going to color $R_b(y) \cup R'_b(y) \setminus \{w_1\}$ using colors in L , where $L = \{1, 2, \dots, d+1\} - \{1, (k+1)\}$. By definition of $R_b(y)$, each vertex u of $R_b(y)$ is adjacent to at most $\lceil \frac{d-1}{2} \rceil - 1$ colored vertices in C , so u is colorable by at least $\lfloor \frac{d+1}{2} \rfloor$ colors of L in a proper coloring. Thus, we can color easily the vertices of $R_b(y)$ by colors of L such that 2 by 2 they get different colors.

There remains a set $L' \subset \{1, \dots, d\}$ (or $\{2, \dots, d\}$) of $|R'_b(y) \setminus w_1|$ colors not used yet.

For each remaining color j , we have $e(C_j, R'_b(y) \setminus \{w_1\}) \leq |R'_b(y) \setminus \{w_1\}| - 2$, and for each $u \in R'_b(y)$, u has at most two colored neighbors in C . Let H be the bipartite graph with bipartition L' and $V = R'_b(y) \setminus \{w_1\}$ such that uj is an edge in H whenever u has no neighbor of color j in G , where $u \in V$ and $j \in L'$. Let $t = |R'_b(y) \setminus \{w_1\}|$. For each $u \in V$, $d_H(u) \geq t - 2$, for each $j \in L'$, $d_H(j) \geq 2$. Thus, by lemma 3.1, there exists a perfect matching in H . Now, if uj is an edge in the matching then color u by j . Finally, we get a dominant vertex y for the color $k + 1$. \square

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